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Multiplicity-free $u_q(n)$ coupling coefficients

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Abstract. The coupling (Wigner–Clebsch–Gordan) coefficients of the unitary quantum algebras $u_q(n)$ are considered. The tensorial properties of the generator powers and their ordered products, used in the explicit projectors and weight lowering procedures, are established. Different expressions for the multiplicity-free isofactors of $u_q(n)$ coupling coefficients (those coupling an arbitrary and symmetric representations) are derived. Explicit expression of the arbitrary isofactors in terms of their boundary values are proposed. Proportionality of the semistretched isofactors to the stretched q-recoupling coefficients of $u_q(n-1)$ (q-analogues of $9j$-symbols) is demonstrated. The stretched isofactors of $u_q(3)$ are expressed in terms of Clebsch–Gordan coefficients of $u_q(2)$.

1. Introduction

The Clebsch–Gordan coefficients of the quantum algebra $u_q(2)$ have been considered by Kirillov and Reshetikhin (1988), Nomura (1989), Ruegg (1990), Groza *et al* (1990), Smirnov *et al* (1991a). The explicit construction of the basis states of irreducible representations (irreps) of the quantum algebra $u_q(n)$ for arbitrary n has been considered by Tolstoy (1990) where the extremal projectors are used and the matrices of the Cartan–Weyl generators in the Gel'fand–Tsetlin basis are presented. The weight lowering and raising operators have also been considered by Ueno *et al* (1989), Quesne (1993), Links and Gould (1993).

Particularly, the $u_q(3)$ algebra, its basis states and some elementary coupling coefficients have been considered by Smirnov and Tolstoy (1990), Smirnov *et al* (1991c), Smirnov and Kharitonov (1993). The q-tensor operators of $u_q(n)$ and the q-analogue of the pattern calculus are introduced by Biedenharn (1990), Biedenharn and Tarlini (1990), Gould and Biedenharn (1992), where the simplest explicit multiplicity-free coupling coefficients of $u_q(n)$ with the extremal operator shift pattern of the corresponding irreps of algebra and its subalgebra are also presented. For this purpose the projectors related to $u_q(n)$ Casimir invariants have been used (Gould *et al* 1992). However, the corresponding reduced coupling coefficients form complete matrices only in the case of the elementary (vector) operators (Gould 1992). Some general properties of the Racah–Wigner algebra for the q-deformed algebras, their coupling and recoupling coefficients have been considered by Lienert and Butler (1992a). The Wigner–Eckart theorem for the unitary quantum groups has been considered by Klimyk (1992). The complementary Hecke algebra technique was used for the simple Clebsch–Gordan coefficients of $u_q(n)$ by Pan and Chen (1993).

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The main purpose of this paper is to consider the more or less general coupling coefficients of quantum algebra $u_q(n)$ and, particularly, their isoscalar factors (isofactors) for the chain $u_q(n) \supset u_q(n-1)$. Using the projectors and weight lowering operators in terms of the ordered generator powers, we develop the results concerning the symmetric tensor operators and corresponding multiplicity-free isofactors of the unitary groups (cf Ališauskas *et al* 1972, Chacón *et al* 1972) as well as the explicit expansion of the more general isofactors in terms of their boundary value (cf Ališauskas 1982, 1983, 1988, Pluhař *et al* 1986).

2. Quantum algebra $u_q(n)$ and symmetric tensor operators of its subalgebra

2.1. Definitions and notations

In order to escape ambiguities, it is necessary to fix the commutation and comultiplication rules for the generators of the unitary quantum algebra $u_q(n) = U_q(u(n))$. The quantum algebra $u_q(n)$ is a deformation of the $u(n)$ enveloping algebra. It is defined by generators e_{ii+1}, e_{i+1i}, $i = 1, 2, \ldots, n-1$, and $h_i = e_{ii}$, $i = 1, 2, \ldots, n$, which satisfy the relations

$$[h_i, h_j] = 0 \qquad i \neq j \tag{2.1a}$$

$$e_{ij+1} = [e_{ij}, e_{jj+1}]_q \equiv e_{ij}e_{jj+1} - q\, e_{jj+1}e_{ij} \qquad i < j \tag{2.1b}$$

$$e_{j+1i} = [e_{j+1j}, e_{ji}]_{q^{-1}} \equiv e_{j+1j}e_{ji} - q^{-1}e_{ji}e_{j+1j} \qquad i < j \tag{2.1c}$$

$$[h_i, e_{jk}] = \delta_{ij}e_{ik} - \delta_{ik}e_{ji} \tag{2.1d}$$

$$[e_{ik}, e_{ki}] = [h_i - h_k] \tag{2.1e}$$

and the Serré identities

$$e_{ik}e_{kl}^2 - [2]e_{kl}e_{ik}e_{kl} + e_{kl}^2e_{ik} = 0 \tag{2.2a}$$

($i < k < l$ or $i > k > l$) which may be written in terms of the q-deformed commutators

$$[[e_{ik}, e_{kl}]_q, e_{kl}]_{q^{-1}} = [[e_{ik}, e_{kl}]_{q^{-1}}e_{kl}]_q = 0\,. \tag{2.2b}$$

Here and below $[x]$ is a q-number

$$[x] = -[-x] = (q^x - q^{-x})/(q - q^{-1})\,. \tag{2.3a}$$

For x integer q-factorial is introduced

$$[x]! = [x][x-1]\ldots[2][1] \tag{2.3b}$$

with $[0]! = 1$ and $[-n]! = \infty$. We use the coproduct rules

$$\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i \tag{2.4a}$$

$$\Delta(e_{ii+1}) = e_{ii+1} \otimes q^{1/2(h_i - h_{i+1})} + q^{-1/2(h_i - h_{i+1})} \otimes e_{ii+1} \tag{2.4b}$$

$$\Delta(e_{i+1i}) = e_{i+1i} \otimes q^{1/2(h_i - h_{i+1})} + q^{-1/2(h_i - h_{i+1})} \otimes e_{i+1i}\,. \tag{2.4c}$$

We denote the irreducible representations (irreps) of $u_q(n)$ by the highest weight $\lambda_{(n)}$ or the partition (Young scheme) $[\lambda_{1n}, \lambda_{2n}, \ldots, \lambda_{nn}]$. The canonical (Gelfand–Tsetlin) basis states may be expressed as (Tolstoy 1990)

$$|\lambda\rangle_q \equiv \left| \begin{array}{ccccccccc} \lambda_{1n} & & \lambda_{2n} & & \cdots & & \lambda_{n-1n} & & \lambda_{nn} \\ & \lambda_{1n-1} & & \lambda_{2n-1} & & \cdots & & \lambda_{n-1n-1} & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & & \lambda_{12} & & \lambda_{22} & & & \\ & & & & \lambda_{11} & & & & \end{array} \right\rangle_q$$

$$= F^{(1)}(\lambda_{(1)}; \lambda_{(2)})F^{(2)}(\lambda_{(2)}; \lambda_{(3)}) \ldots F^{(n-1)}(\lambda_{(n-1)}; \lambda_{(n)}) \left| \begin{array}{c} \lambda_{(n)} \\ \max \end{array} \right\rangle_q \tag{2.5}$$

where $\lambda_{ij}(1 \leqslant i \leqslant j \leqslant n)$ satisfy the standard inequalities

$$\lambda_{ij+1} \geqslant \lambda_{ij} \geqslant \lambda_{i+1,j+1} \tag{2.6}$$

and, in general,

$$\left| \begin{array}{l} \lambda_{(n)} \\ \cdots \\ \lambda_{(k)} \\ \max \end{array} \right\rangle_q \equiv \left| \begin{array}{ccccccccc} \lambda_{1n} & \cdots & & & \cdots & & \cdots & & \lambda_{nn} \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & \lambda_{1k} & & \lambda_{2k} & \cdots & \lambda_{k-1k} & & \lambda_{kk} & \\ & & \lambda_{1k} & & \lambda_{2k} & \cdots & \lambda_{k-1k} & & \\ & & & \cdot & \cdot & \cdot & & & \\ & & & & \lambda_{1k} & & & & \end{array} \right\rangle_q \tag{2.7a}$$

is the highest weight vector of $u_q(k)$:

$$h_j \left| \begin{array}{l} \lambda_{(n)} \\ \cdots \\ \lambda_{(k)} \\ \max \end{array} \right\rangle_q = \lambda_{jk} \left| \begin{array}{l} \lambda_{(n)} \\ \cdots \\ \lambda_{(k)} \\ \max \end{array} \right\rangle_q \qquad j = 1, 2, \ldots, k \tag{2.7b}$$

$$e_{ij} \left| \begin{array}{l} \lambda_{(n)} \\ \cdots \\ \lambda_{(k)} \\ \max \end{array} \right\rangle_q = 0 \qquad \text{if} \quad i < j \leqslant k . \tag{2.7c}$$

In equation (2.5)

$$F^{(k)}(\lambda_{(k)}, \lambda_{(k+1)}) = N^{-1}(\lambda_{(k)}, \lambda_{(k+1)})\; P_{k,q}^{\lambda_{(k)}} \prod_{i=1}^{k} (e_{k+1i})^{\lambda_{ik+1}-\lambda_{ik}} \tag{2.8}$$

is a weight lowering operator with a normalization factor (Tolstoy 1990)

$$N(\lambda_{(k)}; \lambda_{(k+1)}) = \left(\prod_{j=1}^{k} [\lambda_{jk+1} - \lambda_{jk}]! \prod_{1 \leqslant i < j \leqslant k} \frac{[\lambda_{ik} - \lambda_{jk} - i + j]!}{[\lambda_{ik+1} - \lambda_{jk} - i + j]!} \right.$$
$$\left. \times \prod_{1 \leqslant i < j \leqslant k+1} \frac{[\lambda_{ik+1} - \lambda_{jk+1} - i + j - 1]!}{[\lambda_{ik} - \lambda_{jk+1} - i + j - 1]!} \right)^{1/2} \tag{2.9}$$

and an extremal projector of the algebra $u_q(k)$

$$P_{k,q}^{\lambda_{(k)}} = P_{k-1,q}^{\overline{\lambda}_{(k)}} \prod_{i=1}^{k-1} P_{ik} \tag{2.10}$$

with $\overline{\lambda}_{(k)} = [\lambda_{1k}\lambda_{2k} \ldots \lambda_{k-1k}]$ and

$$P_{ij} = \sum_{r_{ij}} (-1)^{r_{ij}} \frac{q^{-(j-i-1)r_{ij}}\; [\lambda_{ik} - \lambda_{jk} - i + j]!}{[r_{ij}]!\,[\lambda_{ik} - \lambda_{jk} - i + j + r_{ij}]!} (e_{ji})^{r_{ij}} (e_{ij})^{r_{ij}} . \tag{2.11}$$

The extremal projectors satisfy conditions

$$e_{ij} P_{k,q}^{\lambda_{(k)}} = P_{k,q}^{\lambda_k} e_{ji} = 0 \qquad 1 \leqslant i < j \leqslant k . \tag{2.12}$$

Note, that the substitution q by q^{-1} and vice versa in (2.1b) and (2.1c) (i.e. going from our and Smirnov *et al* (1991c) case to the Tolstoy (1990) case) is not important for our construction of basis states (2.5) and extremal projectors (2.10), but in the case of the inverse order of P_{ij} chosen in the projector all coefficients $q^{-(j-i-1)r_{ij}}$ should also be inverted (cf Smirnov *et al* 1991c). (Such a problem cannot be disregarded when the lowest indices $i = 1, 2, \ldots$ are eliminated in the restriction to subalgebra processes). The ordering of the separate factors in (2.8) and (2.10) is essential.

2.2. Symmetric tensor operators of subalgebras

Let us introduce the symmetric tensor operators (respectively, covariant and contravariant one) of the subalgebras $u_q(k) \supset u_q(k-1) \supset \ldots \supset u_q(1)$ in terms of generators (2.1):

$$T^{p_k(k,q^{-1})}_{p_{k1},\ldots,p_1} = \left(\frac{[p_k]!}{\prod_{i=1}^{k}[p_i - p_{i-1}]!}\right)^{1/2} \prod_{k\geqslant i\geqslant 1} e_{ik+1}^{p_i-p_{i-1}}\, q^{-1/2\sum_{j=1}^{k}(p_j-p_{j-1})h_j} \tag{2.13}$$

$$T^{\tilde{p}_k(k,q)}_{\tilde{p}_{k-1},\tilde{p}_{k-2},\ldots,\tilde{p}_1} = \left(\frac{[\tilde{p}_k]!}{\prod_{i=1}^{k}[\tilde{p}_i - \tilde{p}_{i-1}]!}\right)^{1/2} q^{1/2\sum_{j=1}^{k}(\tilde{p}_j-\tilde{p}_{j-1})h_j} \prod_{i=1}^{k}((-q)^{k-i}e_{k+1i})^{\tilde{p}_i-\tilde{p}_{i-1}} . \tag{2.14}$$

The transposition formulae

$$e^a_{ik-1}\, e^b_{k-1k} = \sum_s \frac{[a]!\,[b]!}{[s]!\,[a-s]!\,[b-s]!} e^{b-s}_{k-1k}\, e^s_{ik}\, e^{a-s}_{ik-1}\, q^{(a-s)(b-s)} \tag{2.15}$$

$$e^a_{ii+1}\, e^b_{ik} = q^{-ab} e^b_{ik}\, e^a_{ii+1} \tag{2.16}$$

$$e^a_{ii+1}\, e^b_{ki} = \sum_s \frac{(-1)^s\,[a]!\,[b]!}{[s]!\,[a-s]!\,[b-s]!} e^{b-s}_{ki} e^s_{ki+1}\, e^{a-s}_{ii+1}\, q^{-s(h_i-h_{i+1}+a-b+1)} \qquad i+1<k \tag{2.17}$$

and their analogues (see Smirnov *et al* 1991c, appendix) allowed us to prove that the operators (2.13) and (2.14) satisfy the relations of the adjoint action

$$\begin{aligned} e_{ii+1}\, T^{p_k(k,q)}_{p_{k-1},\ldots,p_1} q^{1/2(h_i-h_{i+1})} &- q^{-1+1/2(h_i-h_{i+1})}\, T^{p_k(k,q)}_{p_{k-1},\ldots,p_1} e_{ii+1} \\ &= ([p_{i+1}-p_i][p_i-p_{i-1}+1])^{1/2}\, T^{p_k(k,q)}_{p_{k-1},\ldots,p_i+1,\ldots,p_1} \end{aligned} \tag{2.18a}$$

$$\begin{aligned} e_{i+1i}\, T^{p_k(k,q)}_{p_{k-1},\ldots,p_1} q^{1/2(h_i-h_{i+1})} &- q^{1+1/2(h_i-h_{i+1})}\, T^{p_k(k,q)}_{p_{k-1},\ldots,p_1} e_{i+1i} \\ &= ([p_{i+1}-p_i+1][p_i-p_{i-1}])^{1/2}\, T^{p_k(k,q)}_{p_{k-1},\ldots,p_i-1,\ldots,p_1} \end{aligned} \tag{2.18b}$$

(with q substituted by q^{-1} in the (2.13) case), introduced in order that the Wigner–Eckart theorem could be used (cf Biedenharn and Tarlini 1990, Smirnov *et al* 1991c, Klimyk 1992). The formulae (2.18) may be generalized to the following expansion:

$$T^{p(k,q)}_{\overline{p},0} = \left(\frac{[p-\overline{p}]!\,[\overline{p}]!}{[p]!}\right)^{1/2} \sum_x \frac{(-1)^x}{[x]!\,[\overline{p}-x]!} e^{\overline{p}-x}_{k-1k}\, T^{p(k,q)}_0 e^x_{k-1k}\, q^{1/2\overline{p}(h_{k-1}-h_k)+x(\overline{p}-1/2p-1)} . \tag{2.19}$$

2.3. Matrix elements of the generators powers in the Gelfand–Tsetlin basis

As a rule we take q real and, of course, it is not a root of unity. We will prove the following expression for the matrix elements of the elementary generators of $u_q(n)$ raised to the power p:

$$\begin{aligned} \left\langle \begin{matrix}\lambda\\ \mu'\\ \nu\end{matrix} \middle| e^p_{k-1k} \middle| \begin{matrix}\lambda\\ \mu\\ \nu\end{matrix} \right\rangle_q &= \left\langle \begin{matrix}\lambda\\ \mu\\ \nu\end{matrix} \middle| e^p_{kk-1} \middle| \begin{matrix}\lambda\\ \mu'\\ \nu\end{matrix} \right\rangle_q \\ &= \frac{[p]!\,(d_{k-1}[\mu]\,d_{k-1}[\mu'])^{1/2}\, S_{k,k-1}[\lambda;\mu]\, S_{k-1,k-2}[\mu';\nu]}{S^2_{k-1,k-1}[\mu';\mu]\, S_{k,k-1}[\lambda;\mu']\, S_{k-1,k-2}[\mu;\nu]} \end{aligned} \tag{2.20}$$

where $\lambda = \lambda_{(k)}$, $\mu = \lambda_{(k-1)}$ and $\nu = \lambda_{(k-2)}$ are partitions. We use the notation

$$
\begin{aligned}
{}_lS_{n,m(k')}[\lambda;\mu] &= {}_{-l}S^{-1}_{m,n(-k'-1)}[\mu;\lambda] \\
&= \left(\frac{\prod_{j=1(j\neq -l)}^{m} \prod_{i=1(i\neq l)}^{\min(j+k',n)} [\lambda_i - \mu_j - i + j + k']!}{\prod_{i=1(i\neq l)}^{n} \prod_{j=1(j\neq -l)}^{\min(i-k'-1,m)} [\mu_j - \lambda_i + i - j - k' - 1]!} \right)^{1/2}
\end{aligned}
\tag{2.21}
$$

$$
{}_ld_n = \prod_{1\leqslant i<j\leqslant n, i, j\neq l} [\lambda_i - \lambda_j - i + j] = {}_{\pm l}S^2_{n,n}[\lambda;\lambda] \tag{2.22}
$$

where n, m, l, k' are integers, $-m \leqslant l \leqslant n$. Equal to 0 parameters k' and l usually are omitted. Besides,

$$
{}_nS_{n,m(k)}[\lambda;\mu] = S_{n-1,m(k')}[\overline{\lambda};\mu] \tag{2.23a}
$$

$$
{}_{-m}S_{n,m(k')}[\lambda;\mu] = S_{n,m-1(k')}[\lambda;\overline{\mu}] \tag{2.23b}
$$

where $\overline{\lambda}$, $\overline{\mu}$ are truncated versions of partitions λ, μ.

The matrix elements (2.20) do not depend on the parameters λ_{il} ($l > k$ or $l < k-2$) of $|\lambda\rangle_q$. For $p = 1$ we obtain the usual formula of Jimbo (1986) or Tolstoy (1990). The all dependence of (2.20) from arbitrary ν and λ is included into the square root factors of the numerator and denominator, in accordance with pattern calculus rules (Gould and Biedenharn, 1992) and can be proved by elementary induction from the $p-1$ case to p case. In order to prove the proper normalization of the matrix element (2.20) (which may depend on only μ and μ'), we consider the particular case of (2.20) for $\nu = \overline{\mu} \equiv [\mu_1, \mu_2, \ldots, \mu_{k-2}]$ by means of the following relations:

$$
\left\langle \begin{matrix} \lambda \\ \mu' \\ \overline{\mu} \end{matrix} \middle| e^p_{k-1k} \middle| \begin{matrix} \lambda \\ \mu \\ \overline{\mu} \end{matrix} \right\rangle_q = N^{-1}(\overline{\mu}, \mu') \left\langle \begin{matrix} \lambda \\ \mu' \\ \max \end{matrix} \middle| \prod_{k-2\geqslant i\geqslant 1} e^{\mu'_i - \mu_i}_{ik-1} P^{\overline{\mu}}_{k-2q} e^p_{k-1k} \middle| \begin{matrix} \lambda \\ \mu \\ \max \end{matrix} \right\rangle_q \tag{2.24a}
$$

$$
= N^{-1}(\overline{\mu}, \mu') \left\langle \begin{matrix} \lambda \\ \mu' \\ \max \end{matrix} \middle| \frac{[p]!}{[\mu'_{k-1} - \mu_{k-1}]!} \prod_{k-1\geqslant i\geqslant 1} e^{\mu'_i - \mu_i}_{ik} \middle| \begin{matrix} \lambda \\ \mu \\ \max \end{matrix} \right\rangle_q . \tag{2.24b}
$$

The parameters i are decreasing from the left to the right in the factorization of the separate generators e_{ik} which appeared from the operator $F^{(k-2)}(\overline{\mu}, \mu')$ when acting into the highest weight bra vector. The extremal projector may be omitted in (2.24a) and e_{ik-1} should be transposed with e^p_{k-1k} with the help of (2.15). Thus, formula (2.7c) leaves only a single term in (2.24b), without ambiguity of the intermediate states.

The proper dependence on λ of special and general matrix elements of the generator powers is confirmed once again. Taking into account the symmetry of the $p = 1$ case under permutation of λ_i and ν_{i-1}, we can close the proof of the general case of the matrix element (2.20). The explicit matrix elements (2.20) may be more easy explained when they are factorized into the reduced matrix elements and isofactors of $u_q(k-1) \supset u_q(k-2)$.

3. Isofactors with one of the representations symmetric

3.1. The first class of expressions for isofactors

The pattern calculus rules (Biedenharn 1990, Gould and Biedenharn 1992) may be generalized to the following expression of the special isoscalar factor of $u_q(k) \supset u_q(k-1)$

(taking into account the differences in definition of the coproduct):

$$\begin{bmatrix} \lambda & p & \lambda' \\ \mu & 0 & \mu \end{bmatrix}_q^{(k)} = q^{\mathcal{Q}_0^{(k)}} \frac{([p]! d_k[\lambda'])^{1/2}\, S_{k,k-1}[\lambda';\mu]}{S_{k,k}[\lambda';\lambda]\; S_{k,k-1}[\lambda;\mu]} \tag{3.1}$$

where the q-phase

$$\mathcal{Q}_0^{(k)} = \tfrac{1}{2}\sum_{1\leqslant i<j}^{k} \Delta_i\Delta_j + \tfrac{1}{2}p\left(\sum_{i=1}^{k}\lambda_i - \sum_{j=1}^{k-1}\mu_j\right) - \tfrac{1}{2}\sum_{i=1}^{k}\Delta_i(\lambda_i - i + 1) \tag{3.2a}$$

$$= \tfrac{1}{4}p^2 - \tfrac{1}{4}\sum_{i=1}^{k}\Delta_i(\lambda_i + \lambda_i' - 2i + 2) + \tfrac{1}{2}p\left(\sum_{i=1}^{k}\lambda_i - \sum_{j=1}^{k-1}\mu_j\right) \tag{3.2b}$$

depends on the shifts $\Delta_i = \lambda_i' - \lambda_i$, $p = \sum_{i=1}^{k}\Delta_i$. The dependence of isofactor (3.1) on μ and its q-factor (not included into q-numbers) may be checked by induction from the following relation:

$$U_k(\lambda\ p-1\ \lambda'\ 1; \lambda'^i\ \ p)_q \begin{bmatrix} \lambda & p & \lambda' \\ \mu & 0 & \mu \end{bmatrix}_q^{(k)}$$

$$= \sum \begin{bmatrix} \lambda & p-1 & \lambda'^i \\ \mu & 0 & \mu \end{bmatrix}_q^{(k)} \begin{bmatrix} \lambda'^i & 1 & \lambda' \\ \mu & 0 & \mu \end{bmatrix}_q^{(k)} \begin{bmatrix} p-1 & 1 & p \\ 0 & 0 & 0 \end{bmatrix}_q^{(k)} \tag{3.3}$$

where $\lambda'^i = [\lambda_1', \ldots, \lambda_{i-1}', \lambda_i' - 1, \lambda_{i+1}', \ldots, \lambda_k']$) supposing the absence of the q-phase factor in the q-recoupling (Racah) coefficients. The sum in the RHS is trivial, as well as the last isofactor. We postpone the proof of the proper normalization of (3.1) until subsection 3.2.

Due to formula (2.20) the more general isofactor may be expressed as the reduced matrix element of the unit tensor operator (2.19):

$$\begin{bmatrix} \lambda & p & \lambda' \\ \mu & \overline{p} & \mu' \end{bmatrix}_q^{(k)} = q^{\mathcal{Q}_{\overline{p}}^{(k)}} N_k\begin{pmatrix} \lambda & \lambda' \\ \mu & \mu' \end{pmatrix} ([p-\overline{p}]!\, d_k[\lambda']\, d_{k-1}[\mu])^{1/2}$$
$$\times \sum_{\sigma} (-1)^{\sum_{i=1}^{k-1}(\sigma_i - \mu_i)} q^{-(p-\overline{p}+1)\sum_{j=1}^{k-1}(\sigma_j - \mu_j)}$$
$$\times \frac{d_{k-1}[\sigma]\, S_{k,k-1}^2[\lambda';\sigma]}{S_{k-1,k-1}^2[\sigma;\mu]\, S_{k-1,k-1}^2[\mu';\sigma]\, S_{k,k-1}^2[\lambda;\sigma]} \tag{3.4}$$

where

$$\mathcal{Q}_{\overline{p}}^{(k)} = \tfrac{1}{2}\sum_{1\leqslant i<j}^{k}\Delta_i\Delta_j - \tfrac{1}{2}\sum_{1\leqslant i<j}^{k-1}\overline{\Delta}_i\overline{\Delta}_j + \tfrac{1}{2}\sum_{i=1}^{k-1}\overline{\Delta}_i(\mu_i - i + 1)$$
$$-\tfrac{1}{2}\sum_{j=1}^{k}\Delta_j(\lambda_j - j + 1) + \tfrac{1}{2}(p-\overline{p})\left(\sum_{i=1}^{k}\lambda_i - \sum_{j=1}^{k-1}\mu_j\right) \tag{3.5}$$

is the q-phase; $\Delta_i = \lambda_i - \lambda_i'$ and $\overline{\Delta}_j = \mu_j' - \mu_j$ are the shifts $\left(\overline{p} = \sum_{j=1}^{k-1}\overline{\Delta}_j\right)$;

$$N_k\begin{pmatrix} \lambda & \lambda' \\ \mu & \mu' \end{pmatrix} = \frac{S_{k-1,k-1}[\mu';\mu]\, S_{k,k-1}[\lambda;\mu]}{S_{k,k}[\lambda';\lambda]\, S_{k,k-1}[\lambda';\mu']} \tag{3.6}$$

is the normalization factor and $k-1$ summation parameters $\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_{k-1}]$ accept values in the region

$$\max(\mu_i, \lambda_{i+1}') \leqslant \sigma_i \leqslant \min(\mu_i', \lambda_i) \qquad i = 1, 2, \ldots, k-1\,.$$

3.2. *Proof of the normalization of isofactors*

In order to prove the proper normalization of (3.4) and, thus, (3.1) (which may depend only on λ and λ') we consider isofactor with special values $\mu = \overline{\lambda} \equiv [\lambda_1, \ldots, \lambda_{k-1}]$ and $\mu' = \overline{\lambda'} \equiv [\lambda'_1, \ldots, \lambda'_{k-1}]$. (Note that we use a bar over Greek letters to denote the partitions in order to mark the truncated Young tableaux, but a bar over p or Δ indicates the parameter of the symmetric irrep or shifts of irreps of the subalgebra). In this case $\sigma = \overline{\lambda}$, $\overline{\Delta}_i = \Delta_i (i < k)$, $\Delta_k = p - \overline{p}$ and (3.4) gives

$$\begin{bmatrix} \lambda & p & \lambda' \\ \overline{\lambda} & \overline{p} & \overline{\lambda'} \end{bmatrix}_q^{(k)} = q^{1/2(p-\overline{p})(\overline{p}+k-1)} \times \prod_{i=1}^{k-1} \left(\frac{[\lambda_i - \lambda_k - i + k - 1]! [\lambda'_i - \lambda'_k - i + k]!}{[\lambda_i - \lambda'_k - i + k - 1]! \, [\lambda'_i - \lambda_k - i + k]!} \right)^{1/2} . \tag{3.7}$$

Otherwise we may consider the matrix elements of the extremal projector (2.10)

$$\left\langle \begin{matrix} \lambda_{(k)} \\ \max \end{matrix} \right|_q \left\langle \begin{matrix} p_k \\ \cdots \\ p_1 \end{matrix} \right|_q P_{k,q}^{\lambda'_{(k)}} \left| \begin{matrix} \lambda_{(k)} \\ \max \end{matrix} \right\rangle_q \left| \begin{matrix} p_k \\ \cdots \\ p_1 \end{matrix} \right\rangle_q . \tag{3.8}$$

We use (2.4) and coproduct expansions of the composite generators

$$\Delta(e_{ik}) = e_{ik} \otimes q^{1/2(h_i - h_k)} + q^{-1/2(h_i - h_k)} \otimes e_{ik} - (q - q^{-1}) \sum_{j=i+1}^{k-1} q^{-1/2(h_j - h_k)} e_{ij} \otimes e_{jk} \, q^{1/2(h_i - h_j)} \tag{3.9a}$$

$$\Delta(e_{ki}) = e_{ki} \otimes q^{1/2(h_i - h_k)} + q^{-1/2(h_i - h_k)} \otimes e_{ki} + (q - q^{-1}) \sum_{j=i+1}^{k-1} q^{-1/2(h_i - h_j)} e_{kj} \otimes e_{ji} \, q^{1/2(h_j - h_k)} \tag{3.9b}$$

where $i < j < k$, (cf Smirnov *et al* 1991c) when acting with these generators into $|\lambda_{(k)}\rangle_q \otimes |p_k\rangle_q$. The matrix elements between the highest weight states $\left\langle \begin{smallmatrix} \lambda_{(k)} \\ \max \end{smallmatrix} \right|_q$ and $\left| \begin{smallmatrix} \lambda_{(k)} \\ \max \end{smallmatrix} \right\rangle_q$ do not vanish only for a single term arising from the expansion of each term of the extremal projector (2.10) according to the rules (2.4*b*), (2.4*c*) and (3.9), namely, when all powers of generators e_{ij} or e_{ji} are zero. (Note, that this condition is not satisfied in the case of the inverse order of P_{ij} in the $P_{k,q}^{\lambda_k}$ or after the substitution $q \to q^{-1}$ in definitions (2.1*b*) and (2.1*c*).) The proof of this vanishing begins from terms arising from the factor $\prod_{i=1}^{k-1} P_{ik}$ in (2.10), therefore the matrix elements of the whole projector may be factorized in correspondence with the factorization of the coupling coefficients. Thus, the extremal isofactor (3.7) may be found from the following relation:

$$\begin{bmatrix} \lambda & p & \lambda' \\ \overline{\lambda} & \overline{p} & \overline{\lambda'} \end{bmatrix}_q^{(k)} = \left(\sum_{r_1, \ldots, r_{k-1}} (-1)^{\sum_{i=1}^{k-1} r_i} q^{-\sum_{i=1}^{k-1} (k-i-1) r_i} \times \prod_{i=1}^{k-1} \frac{[\lambda'_i - \lambda'_k - i + k]!}{[r_i]! \, [\lambda'_i - \lambda'_k - i + k + r_i]!} \left\langle \begin{matrix} \lambda \\ \max \end{matrix} \right| q^{-\sum_{i=1}^{k-1} r_i (h_i - h_k)} \left| \begin{matrix} \lambda \\ \max \end{matrix} \right\rangle_q \times \left\langle \begin{matrix} p \\ \overline{p} \\ \cdots \end{matrix} \right| e_{k1}^{r_1} e_{1k}^{r_1} e_{k2}^{r_2} e_{2k}^{r_2} \cdots e_{kk-1}^{r_{k-1}} e_{k-1k}^{r_{k-1}} u \left| \begin{matrix} p \\ \overline{p} \\ \cdots \end{matrix} \right\rangle_q \right)^{1/2} \tag{3.10a}$$

$$= \prod_{i=1}^{k-1} \left(\sum_{r_i} (-1)^{r_i} q^{-r_i(\lambda_i-\lambda_k+k-i-1)} \frac{[\lambda_i'-\lambda_k'-i+k]!}{[r_i]!\,[\lambda_i'-\lambda_k'-i+k+r_i]!} \right.$$
$$\left. \times \frac{[p-\overline{p}]!\,[\lambda_i'-\lambda_i+r_i]!}{[p-\overline{p}-r_i]!\,[\lambda_i'-\lambda_i]!} \right)^{1/2} . \qquad (3.10b)$$

The q-phase factors of the matrix elements of $e_{ki}^{r_i}$ and $e_{ik}^{r_i}$ are mutually inverse in this case and the summation formula

$$\sum_s (-1)^s \frac{q^{s(b-a+c-1)}[a-s]!}{[s]!\,[b-s]!\,[c-s]!} = (-1)^c\, q^{bc} \frac{[a-c]!\,[b-a+c-1]!}{[c]!\,[b]!\,[b-a-1]!} \qquad (3.11)$$

where $b > a$ (Smirnov *et al* 1991a, appendix) applied to (3.10b) gives independently the same expression (3.7) which was derived from (3.4).

3.3. The second class of expressions for isofactors

We see that the expression (3.4) does not simplify for the highest weight state of the resulting irrep. Otherwise due to formulae (2.13), (3.1) and (3.4) and the Wigner–Eckart theorem we write the matrix elements for the products the generator $e_{kk+1}$and $e_{k-1,k+1}$ powers:

$$\left\langle \begin{matrix} \pi \\ \lambda' \\ \mu' \\ \nu \end{matrix} \right| e_{kk+1}^{p-\overline{p}} e_{k-1k+1}^{\overline{p}} \left| \begin{matrix} \pi \\ \lambda \\ \mu \\ \nu \end{matrix} \right\rangle_q = q^{Q_0^{(k)}+\overline{p}(\sum_i \mu_i - 1/2\sum_j \lambda_j - 1/2 \sum \nu_j)}$$
$$\times \frac{([\overline{p}]![p-\overline{p}]!d_k[\lambda])^{1/2}\, S_{k+1,k}[\pi;\lambda]}{S_{k+1,k}[\pi;\lambda']\, S_{k,k}[\lambda';\lambda]} \begin{bmatrix} \lambda & p & \lambda' \\ \mu & \overline{p} & \mu' \end{bmatrix}_{q^{-1}}^{(k)} \begin{bmatrix} \mu & \overline{p} & \mu' \\ \nu & 0 & \nu \end{bmatrix}_{q^{-1}}^{(k-1)} . \qquad (3.12)$$

An alternative expansion of the adjoint action

$$e_{kk+1}^{p-\overline{p}}\, e_{k-1\;k+1}^{\overline{p}} = \sum_y \frac{(-1)^{\overline{p}-y} q^{\overline{p}-y}}{[y]!\,[\overline{p}-y]!} e_{kk+1}^{p-y} e_{k-1k}^{\overline{p}} e_{kk+1}^{y} \qquad (3.13)$$

related to (2.19), together with (2.20) (after substitution $q \to q^{-1}$) leads to other class of expressions for the $u_q(k) \supset u_q(k-1)$ isofactors

$$\begin{bmatrix} \lambda & p & \lambda' \\ \mu & \overline{p} & \mu' \end{bmatrix}_q^{(k)} = q^{Q_{\overline{p}}^{(k)}} N_k^{-1} \begin{pmatrix} \lambda & \lambda' \\ \mu & \mu' \end{pmatrix} \left(\frac{d_k[\lambda']d_{k-1}[\mu]}{[p-\overline{p}]!} \right)^{1/2}$$
$$\times \sum_\sigma (-1)^{\overline{p}-\sum_{i=1}^k(\sigma_i-\lambda_i)} q^{-\overline{p}+\sum_{i=1}^k(\sigma_i-\lambda_i)} \frac{[p-\sum_{i=1}^k(\sigma_i-\lambda_i)]!}{[\overline{p}-\sum_{i=1}^k(\sigma_i-\lambda_i)]!}$$
$$\times \frac{d_k[\sigma]\; S_{k,k-1}^2[\sigma;\mu]}{S_{k,k-1}^2[\sigma;\mu]\; S_{k,k}^2[\sigma;\lambda]\; S_{k,k}^2[\lambda';\sigma]} \qquad (3.14)$$

where notations (2.21), (2.22), (3.5) and (3.6) are used and summation parameters σ_i ($1 \leqslant i \leqslant k$) accept values

$$\max(\mu_i', \lambda_i) \leqslant \sigma_i \leqslant \min(\mu_{i-1}, \lambda_i') .$$

We see that the all summation intervals are different from those of the (3.4) case. We may extend the summation over a chosen parameter σ_i to the all k intervals. The antisymmetric factor $d_k[\sigma]$ ensure vanishing of this additional contribution into the whole quantity (3.14).

Then, the all dependence on σ_i of separate terms may be included into the numerator q-numbers with exception of the denominator factor

$$[\lambda_1' - \sigma_i + i - 1]![\sigma_i - \lambda_k - i + k]!$$

in such a way that the summation formula

$$\sum_s \frac{(-1)^s q^{(n-m-1)s}}{[s]![n-s]!} \prod_{j=1}^{m} [A_j - s] = \delta_{m,n} q^{-n(n+1)/2 + \sum_{j=1}^{m} A_j} \tag{3.15}$$

(where $m \leqslant n$) could be used. Formula (3.15) may be proved by induction with the help of the relation

$$[A - s] = q^{s-n}[A - n] + q^{[A-n]}[n - s]$$

which allows us to reduce m or n without spoiling the condition $m \leqslant n$. The $m = 0$ case of (3.15) (by the way, used together with (2.21) for the proof of (3.13)) should be considered separately. The relation

$$[n - s + 1] = [n + 1]q^s - [s]q^{n+1}$$

leads to

$$\sum_s \frac{(-1)^s q^{(n-1)s}}{[s]!\,[n-s]!} = [n+1] \sum_s \frac{(-1)^s q^{ns}}{[s]!\,[n-s+1]!} - \sum_s \frac{(-1)^s q^{(n-1)s+n+1}}{[s-1]!\,[n-s+1]!}$$

and, after relabelling of the summation parameter, to

$$\sum_s \frac{(-1)^s q^{ns}}{[s]!\,[n+1-s]!} = -\frac{q^n[n]}{[n+1]}(q - q^{-1}) \sum_s \frac{(-1)^s q^{(n-1)s}}{[s]!\,[n-s]!}.$$

The $n = 0$ and $n = 1$ cases are elementary.

Hence, we obtain k different expressions for the $u_q(k) \supset u_q(k-1)$ isofactors:

$$\begin{bmatrix} \lambda & p & \lambda' \\ \mu & \overline{p} & \mu' \end{bmatrix}_q^{(k)} = N_k^{-1} \begin{pmatrix} \lambda & \lambda' \\ \mu & \mu' \end{pmatrix} \left(\frac{d_k[\lambda']\, d_{k-1}[\mu]}{[p - \overline{p}]!} \right)^{1/2}$$
$$\times \sum_{i\sigma} \frac{(-1)^{\varphi_i} q^{R_i^{(k)}}\, {}_i d_k[\sigma]\, {}_i S^2_{k,k-1}[\sigma; \mu]}{{}_i S^2_{k,k-1}[\sigma; \mu']\, {}_i S^2_{k,k}[\sigma; \lambda]\, {}_{-i} S^2_{k,k}[\lambda'; \sigma]} \tag{3.16}$$

where index i is used to denote the omission of the ith row in the partition σ, the summation parameters σ_j are changing in the same intervals as in (3.14) and

$$\varphi_i = \sum_{j=1}^{i-1} (\mu_j' - \mu_j + \lambda_j) + \sum_{j=i+1}^{k} \lambda_j' - \sum_{j=1, j\neq i}^{k} \sigma_j \tag{3.17}$$

$$\begin{aligned} R_i^{(k)} = {} & Q_{\overline{p}}^{(k)} + \tfrac{1}{2}(\lambda_1' - \lambda_k)(\lambda_1' - \lambda_k + 2k - 1) - \overline{p} - \sum_{j=1}^{k-1} (\lambda_j - \lambda_k) \\ & - \sum_{j=1}^{k-1} (\lambda_j - \lambda_{j+1}')\left(\tfrac{1}{2}\lambda_j + \tfrac{1}{2}\lambda_{j+1}' - \lambda_k - j + k - \tfrac{1}{2}\right) \\ & + \tfrac{1}{2}(p - \overline{p})(p - \overline{p} - 1) - \sum_{j=i}^{k-1} \overline{\Delta}_j \left(\mu_j' - \lambda_k - \tfrac{1}{2}\overline{\Delta}_j - j + k - \tfrac{1}{2}\right) \\ & - (p - \overline{p}) \left(\sum_{j=1}^{k} \lambda_j' - \lambda_k - i + k - \sum_{j=1; j\neq i}^{k} \sigma_j \right) \end{aligned} \tag{3.18a}$$

$$= \tfrac{1}{2} \sum_{1\leqslant j<j'}^{k} \Delta_j \Delta_{j'} - \tfrac{1}{2} \sum_{1\leqslant j<j'}^{k-1} \overline{\Delta}_j \overline{\Delta}_{j'} + \tfrac{1}{2} \sum_{j=1}^{k} \Delta_j (\lambda'_j - j) - \tfrac{1}{2} \sum_{j=1}^{k-1} \overline{\Delta}_j (\mu'_j - j)$$
$$- \tfrac{1}{2}(p - \overline{p}) \Big(\sum_{j=1}^{k} \lambda'_j + \sum_{j=1}^{k-1} \mu'_j - 2i + 1 \Big) + (p - \overline{p}) \sum_{j=1;j\neq i}^{k} \sigma_j . \qquad (3.18b)$$

Formula (3.18*a*) has been derived straightforwardly applying (3.15) to (3.14). Afterwards it was rearranged into a more convenient form (3.18*b*).

Equation (3.16) for $k = 2$, $i = 1$ is equivalent to (5.17) of Smirnov *et al* (1991a) (after the substitution $j_1 \leftrightarrow j_2$, $m_1 \leftrightarrow -m_2$), when (4.1) of Groza *et al* (1990) and (3.9) of Nomura (1989) are related to (3.4) (with $k = 2$).

3.4. Some special isofactors and their symmetry

We present here some special isofactors and their symmetry properties which will be used in our next paper. The semistretched case of (3.17) (when $\lambda'_k = \lambda_k$) for $i = 1, 2, \ldots, k-1$ includes $k-2$ summation parameters. We note that in the $q = 1$ case such an isofactor of $u_q(k) \supset u_q(k-1)$ is proportional to the doubly stretched recoupling coefficient (an analogue of $9j$-coefficient—see Ališauskas *et al* 1972).

The completely stretched case of (3.17) (when $\lambda'_1 = \lambda_1 + p$, $\lambda'_j = \lambda_j$, $j = 2, 3, \ldots, k$) for $i = 1$ gives

$$\begin{bmatrix} \lambda & p & \lambda' \\ \mu & \overline{p} & \mu' \end{bmatrix}_q^{(k)} = q^{R_1^{(k)}} \frac{{}_1S_{k,k-1}[\lambda; \mu]}{{}_1S_{k,k-1}[\lambda'; \mu']\, S_{k-1,k-1}[\mu'; \mu]} \left(\frac{[p]!\, d_{k-1}[\mu]}{[p-\overline{p}]!\, d_k[\lambda]} \right.$$
$$\left. \times \prod_{j=2}^{k} \frac{[\lambda_1 - \lambda_j - 1 + j]!}{[\lambda_1 + p - \lambda_j - 2 + j]!} \prod_{j=1}^{k-1} \frac{[\lambda_1 + p - \mu'_j - 1 + j]!}{[\lambda_1 - \mu_j - 1 + j]!} \right)^{1/2} \qquad (3.19)$$

with

$$R_1^{(k)} = \tfrac{1}{2}\overline{p}(\lambda_1 + p - 1) - \tfrac{1}{2} \sum_{1\leqslant i<j}^{k-1} \overline{\Delta}_i \overline{\Delta}_j - \tfrac{1}{2} \sum_{j=1}^{k-1} \overline{\Delta}_j (\mu'_j - j) - \tfrac{1}{2}(p - \overline{p}) \Big(\sum_{j=1}^{k-1} \mu'_j - \sum_{j=2}^{k} \lambda_j \Big)$$

$${}_1S_{k,k-1}[\lambda; \mu] = S^{-1}_{k-1,k-1}[\mu; \lambda^{(-)}] \qquad \lambda^{(-)} = [\lambda_2, \lambda_3, \ldots, \lambda_k] .$$

For the maximal state of the resulting irrep, we take in (3.16) $i = k$ and $\sigma_j = \lambda'_j \equiv \overline{\lambda'_j}$ ($j = 1, 2, \ldots, k-1$) and obtain

$$\begin{bmatrix} \lambda & p & \lambda' \\ \mu & \overline{p} & \overline{\lambda'} \end{bmatrix}_q^{(k)} = (-1)^{\sum_{j=1}^{k-1}(\lambda_j - \mu_j)} q^{R_k^{(k)}} \frac{S_{k-1,k-1}[\overline{\lambda'}; \mu]}{S_{k-1,k}[\overline{\lambda'}; \lambda]\, S_{k,k-1}[\lambda; \mu]}$$
$$\times \left(\frac{d_{k-1}[\mu]}{[p-\overline{p}]!} \prod_{j=1}^{k-1} \frac{[\lambda'_j - \lambda'_k - j + k]!}{[\lambda_j - \lambda'_k - j + k - 1]!} \right)^{1/2} \qquad (3.20)$$

where $R_k^{(k)}$ accepts value

$$R_1^{(k)} = \tfrac{1}{2} \sum_{1\leqslant j<j'}^{k} \Delta_j \Delta_{j'} - \tfrac{1}{2} \sum_{1\leqslant j<j'}^{k-1} \overline{\Delta}_j \overline{\Delta}_{j'}$$
$$- \tfrac{1}{2} \sum_{j=1}^{k-1} (\lambda_j - \mu_j)(\lambda'_j - j) + \tfrac{1}{2}(\lambda'_k - \lambda_k)(\lambda'_k - k) - \tfrac{1}{2}(p - \overline{p})(\lambda'_k - 2k + 1) .$$

We see that (3.7) is also a particular case of (3.20).

For the lowest weight state of the resulting irrep we obtain in (3.4) $\sigma = [\lambda'_2, \lambda'_3, \ldots, \lambda'_n] \equiv \lambda'^{(-)}$ and

$$\begin{bmatrix} \lambda & p & \lambda' \\ \mu & \overline{p} & \lambda'^{(-)} \end{bmatrix}_q^{(k)} = (-1)^{\overline{p}} q^{\tilde{Q}_{(k)}} \left(\frac{[p-\overline{p}]!\, d_{k-1}[\mu]}{\prod_{j=1}^{k}[\lambda'_1 - \lambda_j - 1 + j]!} \right)^{1/2} \times \frac{S_{k,k-1}[\lambda;\mu]}{S_{k-1,k-1}[\lambda'^{(-)};\mu]\; S_{k,k-1}[\lambda;\lambda'^{(-)}]} \tag{3.21}$$

where

$$\tilde{Q}_{(k)} = \tfrac{1}{2}\sum_{i<j} \Delta_i \Delta_j - \tfrac{1}{2}\sum_{i<j} \overline{\Delta}_i \overline{\Delta}_j + \tfrac{1}{2}\sum_{i=1}^{k-1} \overline{\Delta}_i (\mu_i - i + 1) - \tfrac{1}{2}\sum_{i=1}^{k} \Delta_i(\lambda_i - i + 1) + \tfrac{1}{2}(p - \overline{p})(\lambda'_1 - p - \overline{p}) - \overline{p}\,.$$

In future we will also use the following particular case of (3.4):

$$\begin{bmatrix} \lambda & p & \lambda' \\ \overline{\lambda} & 0 & \overline{\lambda} \end{bmatrix}_q^{(k)} = q^{Q_0^{(k)}} \left(\frac{[p]!\, d_k[\lambda'] \prod_{j=1}^{k-1}[\lambda_j - \lambda_k - j + k - 1]!}{d_{k-1}[\lambda] \prod_{j=1}^{k}[\lambda'_j - \lambda_k - j + k]!} \right)^{1/2} \tag{3.22}$$

with

$$Q_0^{(k)} = \tfrac{1}{2}\sum_{i<j} \Delta_i \Delta_j + \tfrac{1}{2} p\, \lambda_k - \tfrac{1}{2}\sum_{j=1}^{k} \Delta_j (\lambda_j - j + 1)\,.$$

Expression (3.4) allows us to prove the following symmetry property:

$$\begin{bmatrix} [-\lambda'_k, -\lambda'_{k-1}, \ldots, -\lambda'_1] & p & [-\lambda_k, \ldots, -\lambda_2, -\lambda_1] \\ [-\mu'_{k-1}, \ldots, -\mu'_1] & \overline{p} & [-\mu_{k-1}, \ldots, -\mu_1] \end{bmatrix}_{q^{-1}}^{(k)} \equiv \begin{bmatrix} \lambda'^{*} & p & \lambda^{*} \\ \mu'^{*} & \overline{p} & \mu^{*} \end{bmatrix}_{q^{-1}}^{(k)} = (-1)^{\overline{p}} q^{\overline{p}k/2 - p(k-1)/2} \left(\frac{d_k[\lambda]\, d_{k-1}[\mu']}{d_k[\lambda']\, d_{k-1}[\mu]} \right)^{1/2} \begin{bmatrix} \lambda & p & \lambda' \\ \mu & \overline{p} & \mu' \end{bmatrix}_q^{(k)}. \tag{3.23}$$

We may go to partitions in notations of the first and resulting irreps in the LHS of (2.23) after addition of $\lambda'_1$to all parameters with minus sign.

In accordance with Klimyk (1993), we may interchange the states to be coupled:

$$\begin{bmatrix} p & \lambda & \lambda' \\ \overline{p} & \mu & \mu' \end{bmatrix}_q^{(k)} = (-1)^{\lambda'_2 - \lambda_2 - \mu'_2 + \mu_2} \begin{bmatrix} \lambda & p & \lambda' \\ \mu & \overline{p} & \mu' \end{bmatrix}_{q^{-1}}^{(k)} \tag{3.24}$$

with fixed the general phase relation and we may apply the *-operation

$$\begin{bmatrix} \lambda & [\dot{0}, -p] & \lambda' \\ \mu & [\dot{0}, -\overline{p}] & \mu' \end{bmatrix}_q^{(k)} = (-1)^{\lambda_{k-1} - \lambda'_{k-1} + \mu_{k-2} - \mu'_{k-2}} \begin{bmatrix} \lambda^{*} & p & \lambda'^{*} \\ \mu^{*} & \overline{p} & \mu'^{*} \end{bmatrix}_{q^{-1}}^{(k)} \tag{3.25}$$

where the phase factor is determined for the $u_q(2)$algebra (subalgebra) and will be chosen as in (3.25), by convention.

It is possible to generalize to the quantum algebra case the Regge-type symmetry of the $U(k)$ isofactors (Ališauskas *et al* 1972) with respect to the summation intervals resticting parameters in (3.4) or (3.16) and the relations and the summation theorems of the multiple hypergeometric series (Holman 1980, Gustafson 1987).

4. Boundary expansion of the general isofactors of $u_q(n) \supset u_q(n-1)$ and the semistretched isofactors

Using the q-binomial expansion of coproduct (Rosso 1988, Smirnov *et al* 1991c) and acting with the operator e^{p}_{nn-1} on the coupled semimaximal state of irrep λ of $u_q(n)$ we may write the following relation between the general and restricted isofactors of $u_q(n) \supset u_q(n-1)$ and $u_q(n-1) \supset u_q(n-2)$ for the arbitrary multiplicity label ρ:

$$\begin{aligned}
&\begin{bmatrix} \lambda' & \lambda'' & \rho & \lambda \\ \mu' & \mu'' & \overline{\rho} & \mu \end{bmatrix}^{n}_{q} \left\langle \begin{matrix} \lambda \\ \mu \\ \nu \end{matrix} \middle| e^{p'+p''}_{nn-1} \middle| \begin{matrix} \lambda \\ \overline{\lambda} \\ \nu \end{matrix} \right\rangle_q \\
&\quad = \sum_{\nu',\nu'',\tilde{\mu}',\tilde{\mu}'',\overline{\overline{\rho}},\tilde{\overline{\rho}}} \frac{[p'+p'']!}{[p']!\,[p'']!}\, q^{Q'_{(n)}} \begin{bmatrix} \mu' & \mu'' & \overline{\rho} & \mu \\ \nu' & \nu'' & \overline{\overline{\rho}} & \nu \end{bmatrix}^{(n-1)}_{q} \\
&\qquad \times \left\langle \begin{matrix} \lambda' \\ \mu' \\ \nu' \end{matrix} \middle| e^{p'}_{nn-1} \middle| \begin{matrix} \lambda' \\ \tilde{\mu}' \\ \nu' \end{matrix} \right\rangle_q \left\langle \begin{matrix} \lambda'' \\ \mu'' \\ \nu'' \end{matrix} \middle| e^{p''}_{nn-1} \middle| \begin{matrix} \lambda'' \\ \tilde{\mu}'' \\ \nu'' \end{matrix} \right\rangle_q \\
&\qquad \times \begin{bmatrix} \lambda' & \lambda'' & \rho & \lambda \\ \tilde{\mu}' & \tilde{\mu}'' & \tilde{\overline{\rho}} & \overline{\lambda} \end{bmatrix}^{(n)}_{q} \begin{bmatrix} \tilde{\mu}' & \tilde{\mu}'' & \tilde{\overline{\rho}} & \overline{\lambda} \\ \nu' & \nu'' & \overline{\overline{\rho}} & \nu \end{bmatrix}^{(n-1)}_{q}
\end{aligned} \tag{4.1}$$

where

$$\overline{\lambda} = [\lambda_1, \lambda_2, \ldots, \lambda_{n-1}]$$

$$p' = \sum_{i=1}^{n-1} (\tilde{\mu}'_i - \mu'_i) \qquad p'' = \sum_{i=1}^{n-1} (\tilde{\mu}''_i - \mu''_i),$$

$$\begin{aligned}
Q'_{(n)} &= \tfrac{1}{2} \sum_{i=1}^{n-1} (\tilde{\mu}'_i - \mu'_i) \Big(2 \sum_{j=1}^{n-1} \tilde{\mu}''_j - \sum_{j=1}^{n} \lambda''_j - \sum_{j=1}^{n-2} \nu''_j \Big) \\
&\quad - \tfrac{1}{2} \sum_{i=1}^{n-1} (\tilde{\mu}''_i - \mu''_i) \Big(2 \sum_{j=1}^{n-1} \tilde{\mu}'_j - \sum_{j=1}^{n} \lambda'_j - \sum_{j=1}^{n-2} \nu'_j \Big)
\end{aligned} \tag{4.2a}$$

$$= \tfrac{1}{2} p' \Big(2 \sum_{j=1}^{n-1} \mu''_j - \sum_{j=1}^{n} \lambda''_j - \sum_{j=1}^{n-2} \nu''_j \Big) - \tfrac{1}{2} p'' \Big(2 \sum_{j=1}^{n-1} \mu'_j - \sum_{j=1}^{n} \lambda'_j - \sum_{j=1}^{n-2} \nu'_j \Big). \tag{4.2b}$$

We may simplify the first or the last isofactor in the RHS of (4.1) choosing $\nu = \overline{\mu} \equiv [\mu_1, \ldots, \mu_{n-2}]$ or $\nu = [\lambda_2, \ldots, \lambda_{n-1}]$. Further, we may consider the structure

$$\begin{aligned}
&\sum_{\nu',\nu'',\overline{\overline{\rho}}} \begin{bmatrix} \mu' & p' & \tilde{\mu}' \\ \nu' & 0 & \nu' \end{bmatrix}^{(n-1)}_{q} \begin{bmatrix} \mu'' & p'' & \tilde{\mu}'' \\ \nu'' & 0 & \nu'' \end{bmatrix}^{(n-1)}_{q} \begin{bmatrix} \tilde{\mu}' & \tilde{\mu}'' & \tilde{\overline{\rho}} & \overline{\lambda} \\ \nu' & \nu'' & \overline{\overline{\rho}} & \nu \end{bmatrix}^{(n-1)}_{q} \\
&\qquad \times \begin{bmatrix} \mu' & \mu'' & \overline{\rho} & \mu \\ \nu' & \nu'' & \overline{\overline{\rho}} & \nu \end{bmatrix}^{(n-1)}_{q} \begin{bmatrix} p' & p'' & p \\ 0 & 0 & 0 \end{bmatrix}^{(n-1)}_{q} q^{p'(\sum_{i=1}^{n-1} \mu''_i - \sum_{i=1}^{n-2} \nu''_i)}
\end{aligned} \tag{4.3a}$$

($p = p' + p''$), extracted from (4.1) according to (2.20) and (3.1) (with the remaining q-factors independent on ν' and ν''). Supposing that the factor $q^{p'(\sum_i \mu''_i - \sum_i \nu''_i)}$ represents the dependence on ν''_i of the diagonal elements of R-matrix in an uncoupled basis

$$(\, \mu'' \quad \nu''; \quad p' \quad 0 \,|\, R^{32}_q \,|\, p' \quad 0; \quad \mu'' \quad \nu'' \,)$$

(this is the case for $u_q(2)$—see Smirnov *et al* 1991b, appendix, and for $u_q(3)$—Lienert and Butler 1992b) we can make a conjecture that structure (4.3*a*) is proportional to the stretched $9j$-symbol and isofactor of $u_q(n-1)$

$$\begin{pmatrix} \mu' & \mu'' & \overline{p} & \mu \\ p' & p'' & & p \\ & & & \\ \tilde{\mu}' & \tilde{\mu}'' & \tilde{\overline{p}} & \overline{\lambda} \end{pmatrix}_q^{(n-1)} \begin{bmatrix} \mu'' & p & \overline{\lambda} \\ \nu & 0 & \nu \end{bmatrix}_q^{(n-1)} \tag{4.3b}$$

(cf the $u_q(2)$ case, Smirnov *et al* 1992).

We also see that in the semistretched case (for $\lambda_n = \lambda'_n + \lambda''_n = 0$), i.e. for those terms in the $u_q(n)$ coproduct which appeared in the $u_q(n-1)$ coproduct decomposition in the case of irreps denoted by the same partitions, the auxiliary isofactor

$$\begin{bmatrix} \lambda' & \lambda'' & \rho & \overline{\lambda} \\ \tilde{\mu}' & \tilde{\mu}'' & \tilde{\overline{\rho}} & \overline{\lambda} \end{bmatrix}_q^{(n-1)} = \delta_{\lambda'\tilde{\mu}'}\delta_{\lambda''\tilde{\mu}''}\delta_{\rho\tilde{\overline{\rho}}} \tag{4.4}$$

and the general semistretched isofactor of $u_q(n)$ should be proportional to the recoupling coefficient of $u_q(n-1)$, in analogy with the relation between the isofactors of $U(n)$ and the recoupling coefficients of $U(n-1)$ (Ališauskas 1969, Ališauskas *et al* 1971, Sullivan 1973, Kramer *et al* 1981).

However, the general q-recoupling coefficients (analogues of $9j$-symbols) are still not defined for $u_q(n)$, $n \geqslant 3$. Since the q-analogue of the $9j$-symbol of $u_q(2)$ (Nomura 1989) in terms of three $6jq$-symbols is not convenient for our purpose, we present here an explicit expression for the semistretched isofactor of $u_q(3) \supset u_q(2)$

$$\begin{aligned}\begin{bmatrix} (a'b') & (a''b'') & (a\ b) \\ (z')i' & (z'')i'' & (z)i \end{bmatrix}_q^{(3)} &= \frac{q^{\tilde{Q}_{(3)}} R[abiz]}{\nabla[i,i',i'']\ R[a'b'i'z']\ R[a''b''i''z'']} \\ &\times \left(\frac{[a+1]!\ [i'-z']!\ [i''-z'']!\ [i+z]!\ [i-z]!\ [v]!}{[a']![a'']![i'+z']![i''+z'']![a'-v]![a''-v]![a+v+1]!} \right)^{1/2} \\ &\times \sum_{r,s} \frac{(-1)^{r+s} q^{-r(a+z-i+1)-s(i+z+1)}[2i'-s]![i-i'+i''+s]!}{[s]![i'+i''-i-s]![r]![v-r]![i-i'-z''+s-r]!} \\ &\times \frac{[a''-r]![a'-v+r]!}{[i'-z'-v-s+r]!}\end{aligned} \tag{4.5}$$

in terms of parameters used by Ališauskas (1988) in the $SU(3)$ case where irreps are labelled as mixed tensor irreps $(a'b')$, $(a''b'')$, $(a\ b)$ and

$$\begin{aligned} &a = \lambda_1 - \lambda_2 \qquad b = \lambda_2 \qquad \lambda_3 = 0 \\ &i = \tfrac{1}{2}(\mu_1 - \mu_2) \qquad z = \lambda_2 - \tfrac{1}{2}(\mu_1 + \mu_2) \\ &a = a' + a'' - 2v \qquad b = b' + b'' + v \qquad z = z' + z'' + v\,. \end{aligned} \tag{4.6}$$

Here the following notations are used:

$$\nabla[abc] = \left(\frac{[a+b-c]![a-b+c]![a+b+c+1]!}{[b+c-a]!} \right)^{1/2} \tag{4.7}$$

$$R[abiz] = \left(\frac{[a+z-i]![a+z+i+1]![b-z-i]![b-z+i+1]!}{[a]![b]![a+b+1]!} \right)^{1/2} \tag{4.8}$$

$$\widetilde{Q}_{(3)} = \tfrac{1}{2}(a' + 2z')(b'' - z'' + i' - i) - \tfrac{1}{2}(a'' + 2z'')(b' - z' - i') + \tfrac{1}{2}(i' + i'' - i)(i - i' + i'' + 1) + \tfrac{1}{2}v(a' - v + 1) - \tfrac{1}{2}a''(i - z) + \tfrac{1}{2}a(i - i' - z'') . \tag{4.9}$$

We see that for $q = 1$ isofactor (4.5) is proportional to the stretched $9j$-symbol (Jucys and Bandzaitis 1977). The completely stretched isofactor of $u_q(3)$ is proportional to the Clebsch–Gordan coefficient of $u_q(2)$:

$$\begin{bmatrix} (a'b') & (a''b'') & (a'+a'', b'+b'') \\ (z')i' & (z'')i'' & (z'+z'')i \end{bmatrix}_q^{(3)} = q^{(a'b''-a''b')/2+z'(a''+b'')-z''(a'+b')} \times \frac{R[a'+a'', b'+b'', i, z'+z'']}{R[a'b'i'z'] \;\; R[a''b''i''z'']} \begin{bmatrix} i' & i'' & i \\ z' & z'' & z \end{bmatrix}_q^{(2)} . \tag{4.10}$$

We see that multiplicity-free couplings for $u_q(3)$ may be exhausted by vanishing of some entrance of array (2.16) of Ališauskas (1988) as well as for $SU(3)$. The usual symmetry properties and substitution (hook permutation) technique allows to derive the remaining multiplicity-free isofactors of $u_q(3)$ (cf. the $SU(3)$ case—Ališauskas 1969). The isofactors of $u_q(n)$ with repeating irreps in the coproduct decomposition will be considered in our next paper, where the above presented results will be used as constructive elements.

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References

Ališauskas S 1969 *Liet. Fiz. Rink.* **9** 641 (in Russian)
——1982 *Liet. Fiz. Rink.* **22** 13 (Engl. transl. 1982 *Sov. Phys.–Coll. Lit. Fiz. Sb.* **22** (2) 9) issue(2)
——1983 *Fiz. Elem. Chastic At. Yadra* **14** 1336 (Engl. transl. 1983 *Sov. J. Part. Nucl.* **14** 563)
——1988 *J. Math. Phys.* **29** 2351
Ališauskas S, Jucys A A and Jucys A P 1972 *J. Math. Phys.* **13** 1329
Ališauskas S, Vanagas V and Jucys A P 1971 *Dokl. AN SSSR* **197** 804
Biedenharn L C 1990 *Quantum Groups (Lecture Notes in Physics 370)* (Berlin: Springer) p 67
Biedenharn L C and Tarlini M 1990 *Lett. Math. Phys.* **20** 271
Chacón E, Ciftan M and Biedenharn L C 1972 *J. Math. Phys.* **13** 577
Gould M D 1992 *J. Math. Phys.* **33** 1023
Gould M D and Biedenharn L C 1992 *J. Math. Phys.* **33** 3613
Gould M D, Links J and Bracken A J 1992 *J. Math. Phys.* **33** 1008
Groza V A, Kachurik I I and Klimyk A U 1990 *J. Math. Phys.* **31** 2769
Gustafson R A 1987 *SIAM J. Math. Anal.* **18** 495
Holman W J III 1980 *SIAM J. Math. Anal.* **11** 523
Jimbo M 1986 *Field theory, Quantum Gravity and Strings (Lecture Notes in Physics 246)* (Berlin: Springer) p 334
Jucys A P and Bandzaitis A A 1977 *Theory of Angular Momentum in Quantum Mechanics* (Vilnius: Mokslas) (in Russian)
Kirillov A N and Reshetikhin N Yu 1988 *Preprint* LOMI E-9-88
Klimyk A U 1992 *J. Phys. A: Math. Gen.* **25** 2919
——1993 Symmetries of Clebsch–Gordan coefficients of the quantum group $U_q(n)$ *Preprint* ITP-93-58E, Kiev
Kramer P, John G and Schenzle D 1981 *Group Theory and the Interaction of Composite Nucleon Systems* (Braunschweig: Vieweg)
Lienert C R and Butler P H 1992a *J. Phys. A: Math. Gen.* **25** 1223
——1992b *J. Phys. A: Math. Gen.* **25** 5577
Links J and Gould M D 1993 *J. Math. Phys.* **34** 1577

Nomura M 1989 *J. Math. Phys.* **30** 2397
Pan F and Chen J-Q 1993 *J. Math. Phys.* **34** 4305, 4316
Pluhař Z, Smirnov Yu F and Tolstoy V N 1986 *J. Phys. A: Math. Gen.* **19** 19
Quesne C 1993 *J. Phys. A: Math. Gen.* **26** 357
Rosso M 1988 *Commun. Math. Phys.* **117** 581
Ruegg H 1990 *J. Math. Phys.* **31** 1085
Smirnov Yu F and Kharitonov Yu I 1993 *Yad. Fiz.* **56** (8) 263
Smirnov Yu F, Tolstoy V N and Kharitonov Yu I 1991a *Yad. Fiz.* **53** 959 (Engl. transl. 1991 *Sov. J. Nucl. Phys.* **53**)
——1991b *Yad. Fiz.* **53** 1746 (Engl. transl. 1991 *Sov. J. Nucl. Phys.* **53**)
——1991c *Yad. Fiz.* **54** 721 (Engl. transl. 1991 *Sov. J. Nucl. Phys.* **54** 437)
——1992 *Yad. Fiz.* **55** 2863 (Engl. transl. 1992 *Sov. J. Nucl. Phys.* **55**)
Sullivan J J 1973 *J. Math. Phys.* **14** 387
Tolstoy V N 1990 *Quantum Groups (Lecture Notes in Physics 370)* (Berlin: Springer) p 118
Ueno K, Takebayashi T and Shibukava Y 1989 *Lett. Math. Phys.* **18** 215